The finite Fourier transform as a vector bundle morphism

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Abstract It is shown that the finite Fourier transform can be viewed as a morphism of the vector bundle of theta functions of fixed weight over the Siegel upper half plane.

Key words: finite Fourier transform, theta functions

§1. The theta view of the finite Fourier transform

At first we recall, in a slightly different notation, the "theta view" of the finite Fourier transform given in $[O]$ It is based on elementary facts about theta functions with characteristics which are explained for instance in $[MF]$, Chapter II, §1, or in $[SH]$. For basic definitions and results about the multidimensional finite Fourier transform we recommend $[DMK]$, chapter 4, 4.4..

For every integer $m > 1$ denote by R_m the ring $(\{0, 1, ..., m-1\}, +, \cdot)$, where $+$ and $\cdot$ denote addition and multiplication modulo m, and for every positive integer g let $V(m,g)$ denote the $\mathbb{C}$-vector space of all functions $f : R_m^g \to \mathbb{C}$. A basis for $V(m,g)$ is given by ihe characteristic functions χ_a, $a \in R_m^g$; $\chi_a(b) := 1$ for $a = b$ and $\chi_a(b) := 0$ otherwise; so the dimension of $V(m,g)$ is m^g. Usually we view g-dimensional vectors as column vector and denote by ${}^tA$ the transpose of a matrix A. With these notations the finite Fourier transform on $V(m,g)$ is the endomorphism

$$F(m,g) : V(m,g) \to V(m,g)$$

defined by

$$F(m,g)(f)(y) := m^{-g/2} \sum_{x \in R_m^g} e^{2\pi i {}^t x y/m} f(x), \;\; f \in V(m,g), \;\; y \in R_m^g \;.$$

It is well known that for all $f \in V(m,g)$ we have $F(m,g)(f) = f^-$ where f^- is defined by $f^-(x) := f(-x)$, $x \in R_m^g$. Hence $F(m,g)$ is a diagonizable automorphism of order 4.

Now denote by $\mathcal{H}_g$ the Siegel upper half plane of degree g; it consists of all complex symmetric $(g \times g)$-matrices Ω whose imaginary part is positive definite.

For every $\Omega \in \mathcal{H}_g$ let $V_m(\Omega)$ denote the $\mathbb{C}-$vector space of all entire complex valued functions on $\mathbb{C}^g$ which are quasiperiodic of weight m with respect to the lattice L_Ω in $\mathbb{C}^g$ which is generated by the columns of the identity matrix 1_g of degree g and by the columns of Ω. Two bases for $V_m(\Omega)$, denoted by

$$(f_{a,\Omega} : \ {}^t a = (a_1, a_2, ..., a_g) \in \mathbb{Z}^g, 0 \le a_i < 1) \text{ resp.}$$

$$(g_{b,\Omega} : \ {}^t b = (b_1, b_2, ..., b_g) \in \mathbb{Z}^g, 0 \le b_i < m) \ ,$$

are obtained from theta functions with characteristics as follows. For all $r, s \in \mathbb{Q}^g$, $z \in \mathbb{C}^g, \Omega \in \mathcal{H}_g$ one defines

$$\vartheta \left[r, s\right](z, \Omega) := \textstyle\sum_{n \in Z^g} \exp(\pi i^t(n+r)\Omega(n+r) + 2\pi i^t(n+r)(z+s))$$

and then

$$f_{a,\Omega}(z) := \vartheta \left[a/m, 0\right](mz, m\Omega),\ {}^t a = (a_1, a_2, ..., a_g) \in \mathbb{Z}^g,\ 0 \le a_i < m,$$

$$g_{b,\Omega}(z) := \vartheta[0, b/m](z, m^{-1}\Omega),\ {}^t b = (b_1, b_2, ..., b_g) \in \mathbb{Z}^g, 0 \le b_i < m.$$

If Ω is fixed we sometimes write $f_a(z) = f_{a,\Omega}(z)$, $g_b(z) = g_{b,\Omega}(z)$.

For every fixed $\Omega \in \mathcal{H}_g$ the assignment $\chi_a \mapsto f_a$, $a \in R_m^g$, yields an isomorphism $V(m,g) \cong V_m(\Omega)$, and under this isomorphism the finite Fourier transform $F(m,g) : V(m,g) \to V(m,g)$ corresponds to a $\mathbb{C}$-linear map

$$F(m, \Omega) : V_m(\Omega) \to V_m(\Omega)$$

such that

$$(1.1) \qquad g_b = m^{g/2} F_m(\Omega)(f_b) \text{ for all } b \in R_m^g.$$

The theta functional equation is proved for instance in $[SH]$. We need the following special case.

$$\vartheta \left[a/m, 0\right] ({}^t((-\Omega)^{-1}z), (-\Omega)^{-1}) =$$

$$= \det(\Omega/i)^{1/2} \exp(\pi i^t z \Omega^{-1} z) \cdot \vartheta \left[0, a/m\right](z, \Omega),$$

which in view of (1.1) can be written as

$$(1.2) \qquad F_m(\Omega)(f_{a,\Omega})(z) =$$

$= m^{-g/2}\det(\Omega/mi)^{-1/2}\exp(-\pi i m^t z\Omega^{-1}z)\cdot f_{a,-\Omega^{-1}}({}^t((-.\Omega^{-1})z)).$

The special case $\Omega = it1_g$, $t \in \mathbb{R}$, $t > 0$ yields the following formula.

(1.3) $\quad F_m(it1_g)(f_{a,it1_g})(z) = t^{-g/2}\exp(-\pi m t^{-1t}zz)\cdot f_{a,it^{-1}1_g}(it^{-1}z)$.

For other ideas and results relating the finite Fourier transform to theta functions see e.g. $[AT]$.

§2. The finite Fourier transform as a vector bundle morphism

The vector bundle which we consider is the holomorphic vector bundle $\mathcal{V}_{m,g}$ of theta functions of fixed weight m and fixed degree g over the Siegel upper half plane $\mathcal{H}_g$, see $[AH]$, 2.3. In in this bundle for every $\Omega \in \mathcal{H}_g$ the corresponding vector space is $V_m(\Omega)$, and a connection for this bundle is given by the "heat equation". Now formulas (1.2) and (1.3) imply the following proposition.

(2.1) Proposition *For fixed m and g the assignment $\mathcal{H}_g \ni \Omega \mapsto F_m(\Omega)$ defines an invertible vector bundle morphism $\mathcal{F}_{m,g} : \mathcal{V}_{m,g} \to \mathcal{V}_{m,g}$ with respect to the map $\mathcal{H}_g \to \mathcal{H}_g$, $\Omega \longmapsto -\Omega^{-1}$, the so called **finite Fourier transform of the vector bundle** $\mathcal{V}_{m,g}$. Restricting the parameters $\Omega \in \mathcal{H}_g$ to the subset $\{it1_g : t \in \mathbb{R}_{>0}\} \subset \mathcal{H}_g$ yields a vector bundle $\mathcal{V}_{m,g,\mathbb{R}}$ over $\mathbb{R}_{>0}$ and a corresponding finite Fourier vector bundle transform $\mathcal{F}_{m,g,\mathbb{R}} : \mathcal{V}_{m,g,\mathbb{R}} \to \mathcal{V}_{m,g,\mathbb{R}}$.*

In the special case $\Omega = it1_g$, $t > 0$, the vector space $V_m(\Omega)$ and the finite Fourier transform $F_m(\Omega)$ decompose as a tensor product

$\otimes^g F_m(it) : \otimes^g V_m(it) \to \otimes^g V_m(it)$.

This follows by an easy computation. We remark that the Poincaré complete reducibility theorem, see e.g. $[MT]$, allows a similar decomposition in a much more general situation. The corresponding vector bundle $\mathcal{V}_{m,g,\mathbb{R}}$ decomposes as $\otimes^g\mathcal{V}_{m,1,\mathbb{R}}$, and the finite Fourier transform $\mathcal{F}_{m,g,\mathbb{R}} : \mathcal{V}_{m,g,\mathbb{R}} \to \mathcal{V}_{m,g,\mathbb{R}}$ decomposes as a g fold tensor product

$\otimes^g\mathcal{F}_{m,1,\mathbb{R}} : \otimes^g\mathcal{V}_{m,1,\mathbb{R}} \to \otimes^g\mathcal{V}_{m,1,\mathbb{R}}$.

Motivated by ideas in $[WZ]$, especially chapters 9, 10, 12, this tensor product form of $\mathcal{F}_{2.g,\mathbb{E}}$ may be viewed to relate time dependent quantum information at time t and at time $1/t$.

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